

Disturbance Models for Offset-Free Model-Predictive Control

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Model predictive control algorithms achieve offset-free control objectives by adding integrating disturbances to the process model. The purpose of these additional disturbances is to lump the plant-model mismatch and/or unmodeled disturbances. Its effectiveness has been proven for particular square cases only. For systems with a number of measured variables (p) greater than the number of manipulated variables (m), it is clear that any controller can track without offset at most m controlled variables. One may think that m integrating disturbances are sufficient to guarantee offset-free control in the m controlled variables. We show this idea is incorrect and present general conditions that allow zero steady-state offset. In particular, a number of integrating disturbances equal to the number of measured variables are shown to be sufficient to guarantee zero offset in the controlled variables. These results apply to square and nonsquare, open-loop stable, integrating and unstable systems.

Introduction

Model predictive control (MPC) arose from the industrial applications called Identification and Command (IDCOM) (Richalet et al., 1978), and Dynamic Matrix Control (DMC) (Cutler and Ramaker, 1979). These control algorithms used finite-impulse or step-response models to predict the future process behavior. In order to obtain offset-free control, the model is updated with feedback information. Comparing the current measured process output and the current predicted output, a constant bias term is added to the future model forecasts. These convolution models cannot be used with open-loop integrating or unstable systems. Generalized predictive control (GPC) (Clarke et al., 1987a,b), instead, used autoregressive models. In order to cover unmodeled disturbances and/or plant-model mismatch, a disturbance model is designed by choosing the so-called “ T polynomial” (see, e.g., Clarke and Mohtadi, 1989), which allows the designer to specify dynamics other than a constant step as in IDCOM or DMC. State-space formulations of MPC were proposed later (Marquis and Broustail, 1988; Kwon and Byun, 1989; Lee et al., 1992; Rawlings and Muske, 1993), which handle open-loop

stable and unstable systems. As shown in Lee et al. (1994), step or impulse response models are particular cases of the state-space models.

In these formulations, offset-free objectives are obtained by augmenting the system state with integrating disturbances. In the original formulations of MPC no analysis was given regarding steady-state offset. In Rawlings et al. (1994) it was shown that a constant output disturbance guarantees offset-free performance for square systems without integrating modes. The constant output disturbance model cannot be used in the presence of integrating modes, because the effects of the plant integrating mode and of the additional disturbance cannot be distinguished. For such cases, double integrated disturbances have been proposed (Lundström et al., 1995). Alternatively, the system can be augmented with integrating modes that affect the system states, as shown in Epler (1997). When the number of measured variables is different from the number of manipulated variables, a natural question arises concerning the number of additional integrating disturbances required to obtain offset-free control. The purpose of this work is to understand steady-state offset with MPC algorithms and provide general design criteria to obtain offset-free performance.

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An alternative or complement to this “design approach” for offset-free performance would be a “disturbance identification” approach, in which one uses plant identification test data to identify which disturbances are nonstationary (integrating) and design internal models for just those disturbances. However, even after such a nonstationary disturbance identification procedure, one would still like to design the additional disturbances that guarantee offset-free performance in case the process disturbances change or the plant–model mismatch changes over time. So the questions addressed in this article are relevant even under a disturbance-identification approach to controller design.

In the following section, we present an example of a system with three measured variables, two of which are controlled by using two manipulated variables. One might think that a reasonable way to remove steady-state offset is to add one integrator to each controlled variable. We show that this choice leads to steady-state offset. Motivated by this and other similar examples, we derive conditions that guarantee zero steady-state offset of the controlled variables when the number of measurements is greater than the number of inputs. Arbitrary linear model dynamics are covered in this work and these results also apply to square systems and nonsquare systems with a number of measurements less than the number of manipulated variables. The rest of the article is organized as follows. In the third section, the MPC formulation with infinite horizon is reviewed. In the fourth section, conditions on the augmented system are derived in order to obtain a stable estimator first, and then offset-free control. Next, in the fifth section, the motivating example is revisited and several controllers are designed according to the results of the previous section. A second example of an ill-conditioned distillation column is also presented to show that the demand of offset-free performance in as many outputs as possible may lead to closed-loop instability in the presence of errors. Finally, in the sixth section, the main accomplishments of this work are summarized. Proofs of the results of the fourth section are reported in the Appendix.

Motivating Example

Plant model

A continuous stirred-tank reactor as shown in Figure 1 is considered. An irreversible, first-order reaction, $A \rightarrow B$, occurs in the liquid phase, and the reactor temperature is regulated with external cooling. This example is taken from Henson and Seborg (1997) with the assumption that the level is not constant. Mass and energy balances lead to the following nonlinear state-space model

$$\frac{dh}{dt} = \frac{F_0 - F}{\pi r^2} \quad (1a)$$

$$\frac{dc}{dt} = \frac{F_0(c_0 - c)}{\pi r^2 h} - k_0 c \exp\left(-\frac{E}{RT}\right) \quad (1b)$$

$$\frac{dT}{dt} = \frac{F_0(T_0 - T)}{\pi r^2 h} + \frac{-\Delta H}{\rho C_p} k_0 c \exp\left(-\frac{E}{RT}\right) + \frac{2Uh}{r\rho C_p}(T_c - T). \quad (1c)$$

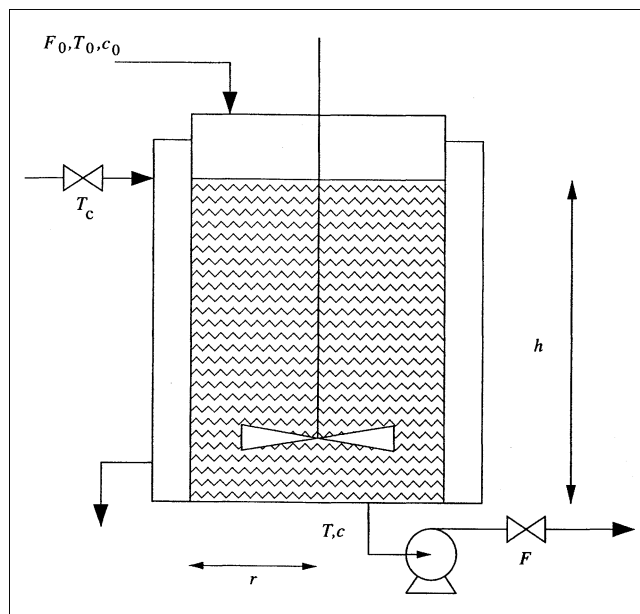


Figure 1. CSTR.

The controlled variables are the level of the tank, h , and the molar concentration of A , c . The third state variable is the reactor temperature, T , while the manipulated variables are the outlet flow rate, F , and the coolant liquid temperature, T_c . Moreover, it is assumed that the inlet flow rate acts as an unmeasured disturbance. The model parameters in nominal conditions are reported in Table 1. The open-loop stable steady-state operating conditions are the following

$$h^s = 0.659 \text{ m}, \quad c^s = 0.877 \text{ mol/L}, \quad T^s = 324.5 \text{ K},$$

$$F^s = 100 \text{ L/min}, \quad T_c^s = 300 \text{ K}.$$

Using a sampling time of 1 min, a linearized discrete state-space model is obtained and, assuming that all the states are measured, the state-space variables are

$$x = \begin{bmatrix} c - c^s \\ T - T^s \\ h - h^s \end{bmatrix}, \quad u = \begin{bmatrix} T_c - T_c^s \\ F - F^s \end{bmatrix}, \quad y = \begin{bmatrix} c - c^s \\ T - T^s \\ h - h^s \end{bmatrix},$$

$$p = F_0 - F_0^s.$$

Table 1. Parameters of the CSTR

Parameter	Nominal Value
F_0	100 L/min
T_0	350 K
c_0	1 mol/L
r	0.219 m
k_0	$7.2 \times 10^{10} \text{ min}^{-1}$
E/R	8,750 K
U	$915.6 \text{ W/m}^2 \cdot \text{K}$
ρ	1 kg/L
C_p	$0.239 \text{ J/g} \cdot \text{K}$
ΔH	$-5 \times 10^4 \text{ J/mol}$

The corresponding linear model is

$$x_{k+1} = Ax_k + Bu_k + B_p p \quad (2a)$$

$$y_k = Cx_k, \quad (2b)$$

in which

$$A = \begin{bmatrix} 0.2511 & -3.368 \times 10^{-3} & -7.056 \times 10^{-4} \\ 11.06 & 0.3296 & -2.545 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} -5.426 \times 10^{-3} & 1.530 \times 10^{-5} \\ 1.297 & 0.1218 \\ 0 & -6.592 \times 10^{-2} \end{bmatrix}$$

$$B_p = \begin{bmatrix} -1.762 \times 10^{-5} \\ 7.784 \times 10^{-2} \\ 6.592 \times 10^{-2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the following, both the linear and the nonlinear models are used to generate the plant response in the presence of disturbances.

Estimator and regulator

The controller is chosen as the solution of the following unconstrained infinite-horizon quadratic optimization problem

$$\min_{u_0, u_1, \dots} \Phi = \sum_{k=0}^{\infty} y_k^T Q y_k + u_k^T R u_k, \quad (3)$$

in which

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

It is clear from the choice of the tuning matrix, Q , that only the composition and the level are required to be at the setpoint (first and third outputs).

Since it is assumed that the disturbance, p , is not measured, the model used by the regulator is

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k. \quad (4)$$

However, it is well known that a disturbance model is required to reject unmeasured nonzero disturbances and obtain offset-free control in the controlled variables. A common choice is the so-called output disturbance model, in which an integrated disturbance is added to each controlled variable. The state-space formulation of this model is

$$x_{k+1} = Ax_k + Bu_k + w_k \quad (5a)$$

$$d_{k+1} = d_k + \xi_k \quad (5b)$$

$$y_k = Cx_k + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} d_k + v_k, \quad (5c)$$

in which $d_k \in \mathbb{R}^2$ is the integrating disturbance, $w_k \in \mathbb{R}^3$ is the state noise, $\xi_k \in \mathbb{R}^2$ is the disturbance noise, and $v_k \in \mathbb{R}^3$ is the output noise. A steady-state Kalman filter is used to estimate the state, x_k , and the disturbance, d_k , given the plant measurements, y_k . The estimator is tuned assuming zero noise for the state (Eq. 5a) and the measurement equation (Eq. 5c), which is the standard DMC-like tuning. Thus, the filtering equations are

$$\begin{bmatrix} \hat{x}_{k|k} \\ \hat{d}_{k|k} \end{bmatrix} = \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{d}_{k|k-1} \end{bmatrix} + \begin{bmatrix} 0 & 1.734 \times 10^{-2} & 0 \\ 0 & 0.9934 & 0 \\ 0 & -8.770 \times 10^{-2} & 0 \\ 1 & -1.734 \times 10^{-2} & 0 \\ 0 & 8.770 \times 10^{-2} & 1 \end{bmatrix} \times \left(y_k - C\hat{x}_{k|k-1} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{d}_{k|k-1} \right). \quad (6)$$

Using the model (Eq. 5) to replace the controlled variables with the system state, the solution of the unconstrained optimization problem (Eq. 3) is

$$u_k = K(\hat{x}_{k|k} - x_t) + u_t, \quad (7)$$

in which $K \in \mathbb{R}^{2 \times 3}$ is the optimal gain matrix computed from the discrete algebraic Riccati equation, and the state and input targets are solutions of the following system:

$$x_t = Ax_t + Bu_t \quad (8a)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \hat{d}_{k|k}. \quad (8b)$$

It is clear from Eqs. 8 that x_t and u_t are the targets that drive the controlled variables to their setpoint (that is, zero).

Disturbance rejection

We assume that at time $t = 10$ min, a disturbance $p = 10$ enters the plant, which is an increment of 10 L/min on the inlet flow rate. The results of the simulation, reported in Figure 2, show offset in the controlled variables. In this work we clarify why offset occurs in cases like this, and we present a general methodology for designing offset-free model predictive controllers.

MPC Algorithm

Preliminary definitions and notations

In this work, we consider linear, time-invariant discrete systems

$$x_{k+1} = Ax_k + Bu_k \quad (9a)$$

$$y_k = Cx_k, \quad (9b)$$

in which $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, with the assumption that (A, B) is stabilizable and (C, A) detectable. If $p > m$, that is, the number of measurements is greater than the number of manipulated variables,

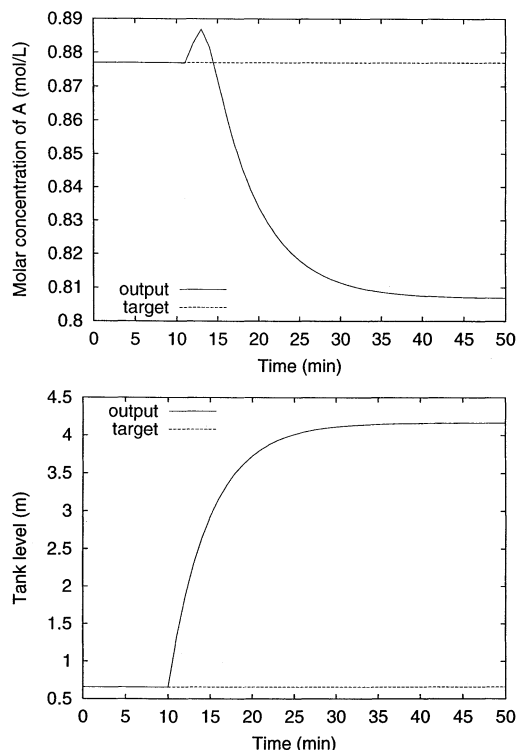


Figure 2. Linearized CSTR.

Disturbance on the inlet flow rate that causes offset.

one cannot attempt to control without offset all the measured variables (see, for example, Corollary 4 of Davison and Smith, 1971). However, one can choose, as controlled variables, m linear combinations of the measured outputs. If $p \leq m$ instead, all the measured variables can be controlled without offset. We treat both cases within the same framework by defining the controlled variable as

$$z = Hy, \quad (10)$$

in which $z \in \mathbb{R}^{n_c}$, $H \in \mathbb{R}^{n_c \times p}$. We assume the following restriction

$$\text{rank} \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} = n + n_c. \quad (11)$$

See Corollary 3 of Davison and Smith (1971) for a continuous-time condition equivalent to this one. As we discuss in the target calculation, this restriction on the original system implies we are able to compensate for the steady-state disturbances in the controlled variables of interest. This condition implies that the number of controlled variables (n_c) cannot exceed either the number of manipulated variables (m) or the number of measurements (p). It also implies that H has to be full row rank, that is, the controlled variables must be independent of each other. Clearly when $m < p$, H defines, as controlled variable, a linear combination of the measurements, while if $p \leq m$, H can be chosen as the identity matrix and all the measured variables can be controlled without offset.

Disturbance model and estimator

In order to achieve offset-free performance we augment the system state with an additional integrating disturbance vector. The roots of this idea are in the works of Davison and coworkers (Davison and Smith, 1971; Davison and Smith, 1974; Qiu and Davison, 1993) and in the internal model principle (Francis and Wonham, 1976). In order to remove offset, one designs a control system that can remove asymptotically constant, nonzero disturbances (Davison and Smith, 1971; Kwakernaak and Sivan, 1972, p. 278). To accomplish this end, the original system is augmented with a replicate of the constant, nonzero disturbance model. Thus the states of the original system are moved onto the manifold that cancels the effect of the disturbance in the controlled variables.

The state and the additional integrating disturbance are estimated from the plant measurement by using a steady-state Kalman filter designed for the following augmented system

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + w_k \quad (12a)$$

$$y_k = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + v_k, \quad (12b)$$

in which $d_k \in \mathbb{R}^{n_d}$, $B_d \in \mathbb{R}^{n \times n_d}$, $C_d \in \mathbb{R}^{p \times n_d}$. The vectors $w_k \in \mathbb{R}^{n+n_d}$ and $v_k \in \mathbb{R}^p$ are zero-mean white-noise disturbances for the augmented state equation and for the output equation, respectively. Thus, the state and the disturbance are estimated as follows

$$\begin{bmatrix} \hat{x}_{k|k} \\ \hat{d}_{k|k} \end{bmatrix} = \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{d}_{k|k-1} \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix} (y_k - C\hat{x}_{k|k-1} - C_d\hat{d}_{k|k-1}), \quad (13a)$$

and the prediction of the future augmented state is obtained by

$$\begin{bmatrix} \hat{x}_{k+1|k} \\ \hat{d}_{k+1|k} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k} \\ \hat{d}_{k|k} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k, \quad (13b)$$

in which $L_x \in \mathbb{R}^{n \times p}$ and $L_d \in \mathbb{R}^{n_d \times p}$ are the filter gain matrices for the state and the disturbance, respectively. Equations 13a and 13b can be condensed into one step, known as Kalman predictor

$$\begin{bmatrix} \hat{x}_{k+1|k} \\ \hat{d}_{k+1|k} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{d}_{k|k-1} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (y_k - C\hat{x}_{k|k-1} - C_d\hat{d}_{k|k-1}), \quad (14)$$

in which $L_1 \in \mathbb{R}^{n \times p}$ and $L_2 \in \mathbb{R}^{n_d \times p}$ are the predictor gain matrices for the state and the disturbance, respectively. Straightforward algebraic calculations show

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} L_x \\ L_d \end{bmatrix}. \quad (15)$$

For convenience, we introduce the following notation

$$\tilde{A} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}, \quad \tilde{C} = [C \quad C_d], \quad \tilde{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

Thus, the predictor gain matrix is computed from the following

$$\tilde{L} = \tilde{A}\Sigma\tilde{C}^T(\tilde{C}\Sigma\tilde{C}^T + R_v)^{-1}, \quad (16)$$

in which $R_v \in \mathbb{R}^{p \times p}$ is a symmetric positive definite matrix associated with the output noise, v_k , and the matrix $\Sigma \in \mathbb{R}^{(n+n_d) \times (n+n_d)}$ is the unique symmetric positive semidefinite solution of the discrete algebraic Riccati equation

$$\Sigma = \tilde{A}\Sigma\tilde{A}^T + \tilde{Q}_w - \tilde{A}\Sigma\tilde{C}^T(\tilde{C}\Sigma\tilde{C}^T + R_v)^{-1}\tilde{C}\Sigma\tilde{A}^T. \quad (17)$$

The matrix $\tilde{Q}_w \in \mathbb{R}^{(n+n_d) \times (n+n_d)}$ is the symmetric positive semidefinite associated with the augmented state noise w_k .

It is important to notice that the additional disturbances, d , are not controllable by the inputs u . However, since they are observable, we use their estimates to remove their influence from the controlled variables, as discussed in the next section.

Target calculation and regulator

Given the current estimate of the disturbance, $\hat{d}_{k|k}$, the state and input target are computed by solving the following quadratic program (Muske and Rawlings, 1993)

$$\min_{x_t, u_t} (u_t - \bar{u})^T \tilde{R}(u_t - \bar{u}), \quad (18a)$$

subject to

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} B_d \hat{d}_{k|k} \\ -HC_d \hat{d}_{k|k} + \bar{z} \end{bmatrix} \quad (18b)$$

$$Eu_t \leq e \quad (18b)$$

$$F(Cx_t + C_d \hat{d}_{k|k}) \leq f, \quad (18d)$$

in which $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and $\tilde{R} \in \mathbb{R}^{m \times m}$ are symmetric positive definite, and $\bar{z} \in \mathbb{R}^{n_c}$ and $\bar{u} \in \mathbb{R}^m$ are the set points for the controlled and manipulated variables, respectively. Notice that, often, the input set point is not specified and it can be assumed zero in order to use, if possible, the optimal input in a least-squares sense. $E \in \mathbb{R}^{\nu \times m}$ and $e \in \mathbb{R}^\nu$ describe linear input constraints, while $F \in \mathbb{R}^{\xi \times p}$ and $f \in \mathbb{R}^\xi$ describe linear output constraints, in which ν and ξ are the number of input and output constraints, respectively. It is assumed that the feasible region of Eq. 18 is nonempty. Notice that Eq. 11 implies that the unconstrained target calculation (that is, Eq. 18 subject to Eq. 18b only) has a solution for all $\hat{d}_{k|k}$ and \bar{z} . If the feasible region of Eq. 18 is empty, input and output constraints are too stringent, and the n_c -controlled variables cannot be tracked to their setpoint without offset. In this case, one can solve a different quadratic pro-

gram that minimizes the offset of the controlled variables in a least-squares sense (Muske and Rawlings, 1993).

Next, the input sequence is computed as the solution of the following infinite-horizon optimization problem

$$\min_{u_0, u_1, \dots} \Phi = \sum_{k=0}^{\infty} (z_k - \bar{z})^T \bar{Q}(z_k - \bar{z}) + (u_k - u_t)^T R(u_k - u_t), \quad (19a)$$

subject to the model (Eq. 12) and to input and output constraints

$$Eu_k \leq e \quad k = 0, 1, \dots \quad (19b)$$

$$Fy_k \leq f \quad k = 0, 1, \dots \quad (19c)$$

The penalty matrices $R \in \mathbb{R}^{m \times m}$ and $\bar{Q} \in \mathbb{R}^{n_c \times n_c}$ are assumed to be symmetric and positive definite.

Using the targets computed from Eq. 18, we introduce the following deviation variables

$$w_j = \hat{x}_{k+j|k} - x_t, \quad v_j = u_{k+j} - u_t. \quad (20)$$

Thus the regulator optimization problem (Eq. 19) becomes

$$\min_{v_0, v_1, \dots} \Phi = \sum_{j=0}^{\infty} w_j^T Q w_j + v_j^T R v_j \quad (21a)$$

subject to

$$w_{j+1} = Aw_j + Bv_j \quad k = 0, 1, \dots \quad (21b)$$

$$Ev_j \leq e - Eu_t \quad k = 0, 1, \dots \quad (21c)$$

$$FCw_j \leq f - F(Cx_t + C_d \hat{d}_{k|k}) \quad k = 0, 1, \dots, \quad (21d)$$

in which $Q = C^T H^T \bar{Q} H C \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite. We assume that

$$(Q^{1/2}, A) \quad (22)$$

is detectable. It can be shown that this condition is satisfied if and only if (HC, A) is detectable. This condition forces the unstable modes of A to be “seen” in the regulator objective function.

The solution of the infinite-horizon constrained problem (Eq. 21) is discussed in Chmielewski and Manousiouthakis (1996) and Scokaert and Rawlings (1998), in which Eq. 21 is reparameterized in terms of a finite number of decision variables and constraints by appending the optimal LQR unconstrained control law, $v_j = Kw_j$, at the end of a finite horizon, N .

Main Results

Restrictions on the augmented system

The disturbance model is defined by choosing the matrices B_d and C_d . More complex disturbance models requiring identification from data can be considered, but (B_d, C_d) allow enough flexibility to meet most process control objectives

without requiring disturbance model identification procedures. Since the additional modes introduced by the disturbance are unstable, it is necessary to check the detectability of the augmented system. Detectability of the augmented system is a necessary and sufficient condition for a stable estimator to exist, that is, such that $\tilde{A} - \tilde{L}\tilde{C}$ is stable [see, e.g., Sontag (1998)]. We have the following results.

Lemma 1 (Detectability of the Augmented System). The augmented system (Eq. 12) is detectable if and only if the nonaugmented system (Eq. 9) is detectable, and the following condition holds

$$\text{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d. \quad (23)$$

Corollary 1 (Dimension of the Disturbance). The maximum dimension of the disturbance d in Eq. 12 such that the augmented system is detectable is equal to the number of measurements, that is,

$$n_d \leq p. \quad (24)$$

A pair of matrices (B_d, C_d) such that Eq. 23 is satisfied always exists. In fact, since (C, A) is detectable, the submatrix $\begin{bmatrix} I - A \\ C \end{bmatrix} \in \mathbb{R}^{(p+n) \times n}$ has rank n . Thus, we can choose any $n_d \leq p$ columns in \mathbb{R}^{p+n} independent of $\begin{bmatrix} I - A \\ C \end{bmatrix}$ and build B_d and C_d accordingly. Clearly, this methodology is not intended for design, but just to show that a detectable augmented system can always be constructed. Design methods for disturbance modeling are currently under investigation (Pannocchia, 2002).

The continuous-time versions of Lemma 1 and Corollary 1 were first stated and proved by Smith and Davison (1972, p. 1214). Morari and Stephanopoulos (1980) provide a nice, compact proof of the continuous-time version using the Hautus lemma. Similar conditions have been found useful in other contexts as well (Kurtz and Henson, 1998).

Offset-free disturbance models

The results of the previous section provide a tool for checking whether or not the augmented system is detectable and therefore whether a stable estimator exists. Once the disturbance model matrices (B_d, C_d) are chosen such that Eq. 23 holds, the augmented system is detectable. In the motivating example, it is easy to verify that this condition holds. Nevertheless, the closed-loop response shows offset in the controlled variable. The next results provide sufficient conditions for zero offset in the controlled variables.

Lemma 2 (Zero Offset). Consider a system controlled by the MPC algorithm, as described in the third section. The target problem (Eq. 18) is assumed feasible. Assume that the closed-loop system reaches a steady state with output y_s and input u_s , and suppose that input and output constraints are not active at this steady state. Let $K_e \in \mathbb{R}^{n_c \times p}$ be the follow-

$$K_e = H \left[C(I - A - BK)^{-1} (A + BK) L_x + I \right],$$

in which $K \in \mathbb{R}^{m \times n}$ is the optimal controller gain computed from the Riccati equation. If

$$\text{null}(L_d) \subseteq \text{null}(K_e), \quad (25)$$

there is zero offset in the controlled variable, that is,

$$Hy_s = \bar{z}. \quad (26)$$

Proof. See the Appendix for proof of Lemma 2.

When $\text{null}(L_d) \not\subseteq \text{null}(K_e)$, it is easy to construct a plant such that the closed-loop system is stable and there is steady-state offset. Therefore, the condition stated in the previous lemma is necessary to guarantee zero offset in the presence of arbitrary plant-model mismatch. If L_d is square and full rank, the assumption, $\text{null}(L_d) \subseteq \text{null}(K_e)$, is satisfied independently of K_e . When the number of integrating disturbances is less than the number of measurements (i.e., $n_d < p$), L_d is not square, and Eq. 25 must be checked to guarantee zero offset in the presence of plant-model mismatch. A second problem related to choosing $n_d < p$ is that, in general, the structure of K_e also depends on the regulator tuning, and therefore Eq. 25 may not hold if the tuning parameter of the controller change.

There are particular cases in which offset-free performance can be guaranteed by adding a number of disturbances equal to the number of controlled variables ($n_d = n_c < p$). A class of examples can be described as follows. Let the system state be partitioned as $x = [x_1^T \ x_2^T]^T$, the output as $y = [y_1^T \ y_2^T]^T$, and let the controlled variable be $z = y_1$. Let the model and disturbance model matrices be as follows (each term is a matrix of appropriate dimensions)

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_d = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

in which A_{11} and A_{22} are strictly stable square matrices. The regular for tuning matrix is $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$ and stability of A_{11} and A_{22} implies that $(Q^{1/2}, A)$ is detectable. Given any positive definite matrix, R , the optimal controller gain matrix has the following structure

$$K = [\times \ 0],$$

which implies that $A + BK$ is an upper-block triangular matrix. If there is zero noise related to x_1 in the state equation and A_{11} is stable, it can be shown that the filter structure is

$$L_x = \begin{bmatrix} 0 & 0 \\ 0 & \times \end{bmatrix}, \quad L_d = [I \ 0].$$

Given the structure of H , C , $A + BK$, and L_x , straightforward block matrix calculations show that $K_e = [I \ 0] = L_d$, and therefore Eq. 25 holds.

However, we are mainly interested in conditions for zero offset that hold for generic linear systems and disturbance models. We have the following results.

Lemma 3 (Full Rank of L_d). Given disturbance model matrices $B_d \in \mathbb{R}^{n \times n_d}$, $C_d \in \mathbb{R}^{p \times n_d}$ such that the assumptions of Lemma 1 are satisfied, and

$$n_d = p, \quad (27)$$

the disturbance filter gain matrix in Eq. 13a, satisfies

$$L_d \in \mathbb{R}^{p \times p} \quad (28)$$

$$\text{rank } L_d = p. \quad (29)$$

Proof. See the Appendix for proof of Lemma 3.

Theorem 1 (Main Result). Consider a system controlled by the MPC algorithm as described in the third section. The target problem (Eq. 18) is assumed feasible. Augment the system model with a number of integrating disturbances equal to the number of measurements ($n_d = p$), choose any $B_d \in \mathbb{R}^{n \times p}$, $C_d \in \mathbb{R}^{p \times p}$, such that

$$\text{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + p.$$

If the closed-loop system is stable and constraints are not active at steady state, there is zero offset in the controlled variables, that is,

$$Hy_s = \bar{z}, \quad (30)$$

in which y_s is the plant output.

Proof. See the Appendix for proof of Theorem 1.

Remarks

It is important to emphasize that, in proving the previous results, no assumptions have been made on the plant dynamics. That is, these results apply even when the plant does not satisfy the model (Eq. 9). In other words, if the system reaches a steady state without active constraints, there is no offset in the controlled variables regardless of the plant dynamics. If constraints were active at steady state, for example, one or more inputs saturate at steady state, in general it is impossible to achieve offset-free control. This inability, which belongs to any kind of control strategy and not specifically to MPC, is due to the fact that input constraints are too stringent for the controlled variables to reach their setpoint. However, the results presented do not depend on the possibility of transient constraint saturation, as shown in the fifth section.

In this work the disturbance model chosen (Eq. 12) is an integrated disturbance that enters through the model dynamic matrix, A . If desired, one can choose different dynamics for the disturbance, as in the following augmented system:

$$\begin{bmatrix} x_{k+1} \\ p_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_p & 0 \\ 0 & A_p & B_d \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ p_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u_k \quad (31a)$$

$$y_k = \begin{bmatrix} C & C_p & 0 \end{bmatrix} \begin{bmatrix} x_k \\ p_k \\ d_k \end{bmatrix}, \quad (31b)$$

in which $p_k \in \mathbb{R}^{n_p}$, $B_p \in \mathbb{R}^{n \times n_p}$, $C_p \in \mathbb{R}^{p \times n_p}$, $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_d \in \mathbb{R}^{n_p \times n_d}$. Assuming that the disturbance dynamic matrix, A_p , is strictly stable, all the results developed can be extended with minor changes. In particular, in Lemma 1 the condition to check in place of Eq. 23 becomes

$$\text{rank} \begin{bmatrix} I - A & -B_p & 0 \\ 0 & I - A_p & -B_d \\ C & C_p & 0 \end{bmatrix} = n + n_p + n_d. \quad (32)$$

As a corollary, we again obtain that the maximum number of integrated disturbances that can be added without losing detectability is equal to the number of measurements, that is, $n_d \leq p$. If we choose $n_d = p$ and the closed-loop system is stable, there is zero steady-state offset in the controlled variable, that is, $Hy_s = \bar{z}$.

The results shown here can be extended to finite (control and/or prediction) -horizon MPC, provided that the corresponding unconstrained control law is nominally stabilizing, that is, it corresponds to a linear feedback gain matrix, K , that makes the matrix $A + BK$ strictly stable. This is typically accomplished in finite-horizon MPC by choosing the output and input penalty matrices and the horizon length.

Case Studies

Revisiting the motivating example

By applying Lemma 4.1 to the motivating example, we can verify that the augmented system in Eq. 5 is detectable, and therefore the Kalman filter used in Eq. 6 results in a stable estimator. However, the filter gain matrix corresponding to the integrated disturbance is

$$L_d = \begin{bmatrix} 1 & -1.734 \times 10^{-2} & 0 \\ 0 & 8.770 \times 10^{-2} & 1 \end{bmatrix},$$

which has rank equal to two and, therefore, has a null space. The term $e_k = y_k - Cx_k - C_d d_k$ does not go to zero, while the product $L_d e_k$ does. We can verify that $\text{null}(L_d) \not\subseteq \text{null}(K_e)$ and that $e_s = \lim_{k \rightarrow \infty} e_k$ is not in the null space of K_e . Therefore, there is a steady-state offset in the controlled variables.

The results of the previous section provide conditions for obtaining zero offset in the controlled variables, c and h . Lemma 1 states a detectability condition on the augmented system, given the disturbance model matrices, B_d and C_d , while Corollary 1 shows that the maximum number of integrated disturbances that can be added is three. Finally, Theorem 1 states that if three integrated disturbances are added in such a way that the augmented system is detectable, there is a guarantee of offset-free control.

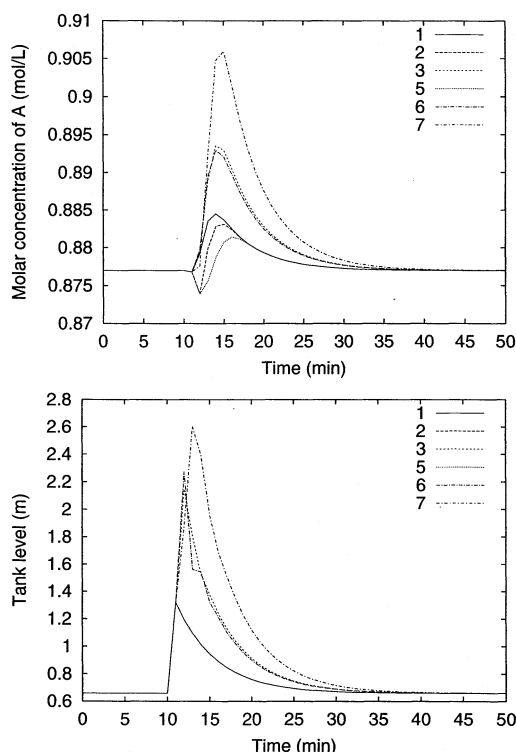
In the motivating example we added two integrated disturbances to two of the three measured variables. One can try to add three disturbances to all three measured variables, but in this case the augmented system is not detectable due to the presence of the integrating mode in the process model (tank level). Other possibilities are adding two integrated disturbances to the manipulated variables and one to a measured variable, and vice versa. In six cases the augmented system is detectable; in other cases it is not, as shown in Table 2. It is

Table 2. Detectability of Disturbance Models

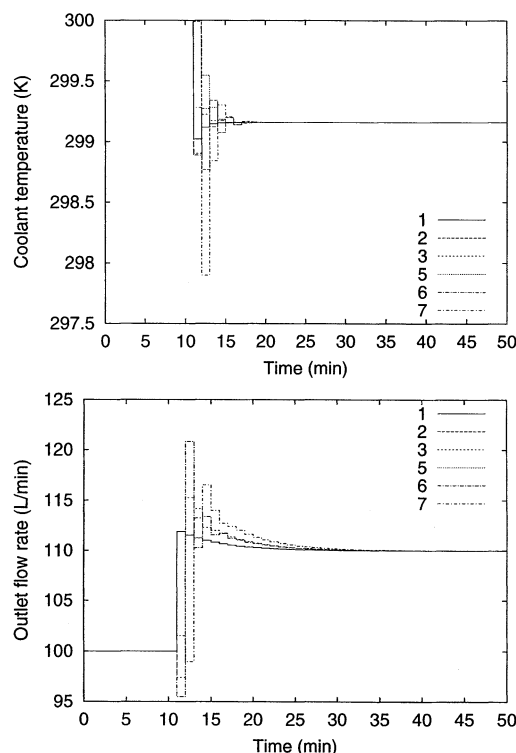
ID	Dist. on Inputs	Dist. on Outputs	Detectable
0	—	c, T, h	No
1	T_c, F	c	Yes
2	T_c, F	T	Yes
3	T_c, F	h	Yes
4	T_c	Any	No
5	F	c, T	Yes
6	F	c, h	Yes
7	F	T, h	Yes

interesting to note that, if the integrated disturbance is added to the manipulated variable T_c , but not to F , the augmented system is not detectable. This property comes from the fact that T_c does not influence the integrating mode associated with h .

The results of simulation, obtained with all detectable disturbance models reported in Table 2, are shown in Figures 3 and 4. For each augmented system, a steady-state Kalman filter has been designed assuming zero noise on the state equation and arbitrarily small noise on the measurement equation. For each disturbance model, an unconstrained controller has been designed with the same tuning specified in the second section. As expected, all controllers achieve offset-free control of the composition and the level. However, the robust performance of these offset-free controllers is quite different. For example, disturbance models #1, #2, and #5 show better disturbance rejection and require smaller variations of the inputs than the other ones. This behavior can be associated with the fact that, in the other disturbance mod-

**Figure 3. Linearized CSTR.**

Rejection of the disturbance on the inlet flow rate: outputs.

**Figure 4. Linearized CSTR.**

Rejection of the disturbance on the inlet flow rate: inputs.

els, an integrator is added to the third measurements, that is, the level, which already has one integrating mode. These simulations have been repeated using the nonlinear model (Eq. 1) as the plant, and the results are shown in Figures 5 and 6. In these nonlinear plant simulations we included the following input constraints,

$$299 \text{ K} \leq T_c \leq 301 \text{ K}, \quad 85 \text{ L/min} \leq F \leq 115 \text{ L/min}, \quad (33)$$

in order to show that transient constraint saturations do not eliminate the offset-free properties of the proposed design. As stated in Theorem 1, these disturbance models lead to zero steady-state offset even if constraints (not active at steady state) are present and/or the plant is nonlinear. However, the difference in robustness of these disturbance models is emphasized by the nonlinear behavior of the plant.

Offset-free control: The other side of the coin

In this section we emphasize some risks of offset-free control. We consider the following transfer-function model of an ill-conditioned distillation column (Morari and Zafiriou, 1989)

$$G(s) = \frac{1}{75s + 1} \begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix}. \quad (34)$$

We assume that the plant is given by the following transfer function

$$G_p(s) = \frac{1}{75s + 1} \begin{bmatrix} 0.878 & -0.880 \\ 1.100 & -1.096 \end{bmatrix}. \quad (35)$$

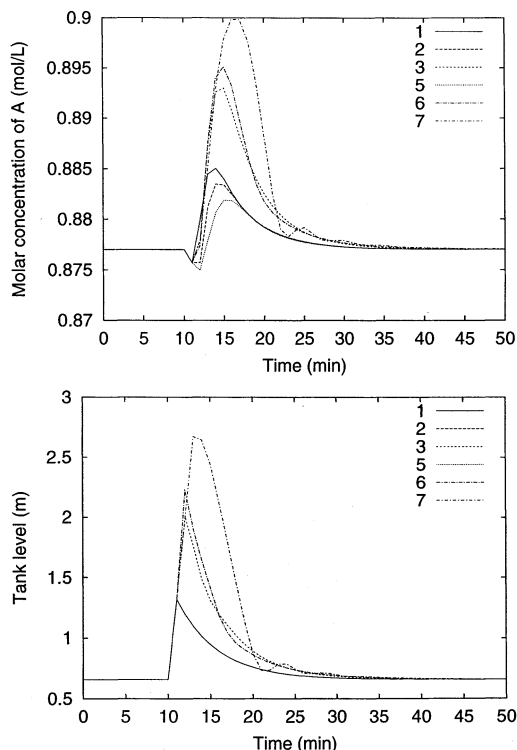


Figure 5. Nonlinear CSTR constrained.

Rejection of the disturbance on the inlet flow rate: outputs.

The determinants of the gain matrices of $G(s)$ and $G_p(s)$ have opposite signs, and it is expected that any feedback controller with integral action designed for $G(s)$ would lead to instability if the plant is $G_p(s)$ [see, for example, Skogestad and Postlethwaite (1996)].

Assuming a sampling time of 5 min, a discrete state-space minimal realization of $G(s)$ has two states, two inputs, and two outputs. The state-space matrices are

$$A = \begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}, \quad B = (1 - \phi) \begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

in which $\phi = \exp(-5/75)$.

Three controllers are compared:

- **MPC 1.** In this controller, no integrating disturbances are added and a steady-state Kalman filter is designed for the nonaugmented system, assuming zero noise of the state equation and arbitrarily small noise for the output equation. The controller tuning matrices as in Eq. 21 are

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which means that both outputs are required to reach setpoint.

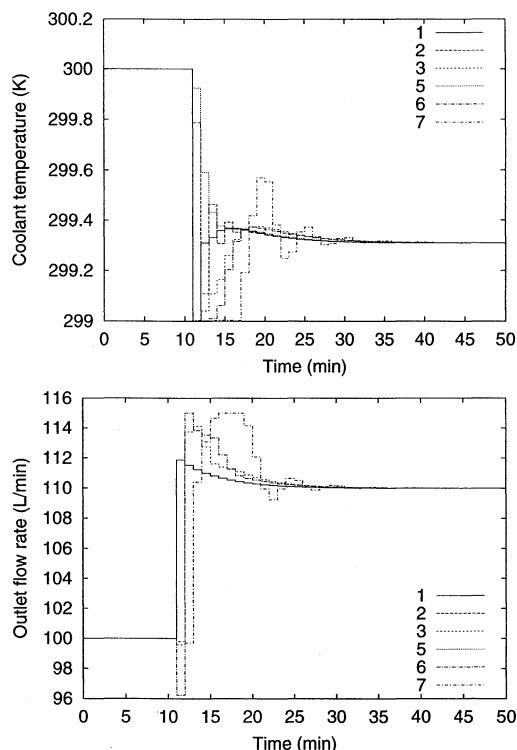


Figure 6. Nonlinear CSTR constrained.

Rejection of the disturbance on the inlet flow rate: inputs.

- **MPC 2.** In this controller, two integrating disturbances are added to the process outputs:

$$B_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and a steady-state Kalman filter is designed for the augmented system, assuming nonzero noise for the disturbance equation, zero noise for the state equation, and arbitrarily small noise for the output equation. The controller tuning matrices are the same as in MPC 1.

- **MPC 3.** In this controller, two integrating disturbances are still added to the process outputs and the same Kalman filter is used. However, we require that only the first output reaches its setpoint by choosing the following controller tuning matrices,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or, in other words, the controlled variable is $z = [1 \ 0]y$.

Simulations of a setpoint change from the origin to $\bar{y} = [1 \ 0]^T$ using the three controllers are reported in Figures 7 and 8. As expected, MPC 2 leads to an unstable closed-loop system, because it is designed to remove offset in both outputs. MPC 1 is not designed for zero steady-state offset in the presence of modeling error and leads to a stable closed-loop system, but with offset in the controlled variables. MPC 3 is designed for removing offset in the first output and leads to a stable closed-loop system. From Theorem 1, we know that if the closed-loop system is stable, there is zero offset in

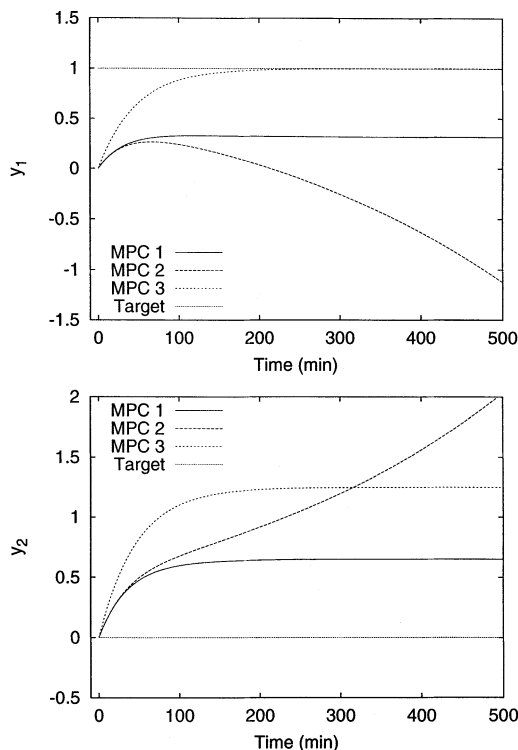


Figure 7. Ill-conditioned column outputs.
Setpoint change in the presence of gain errors.

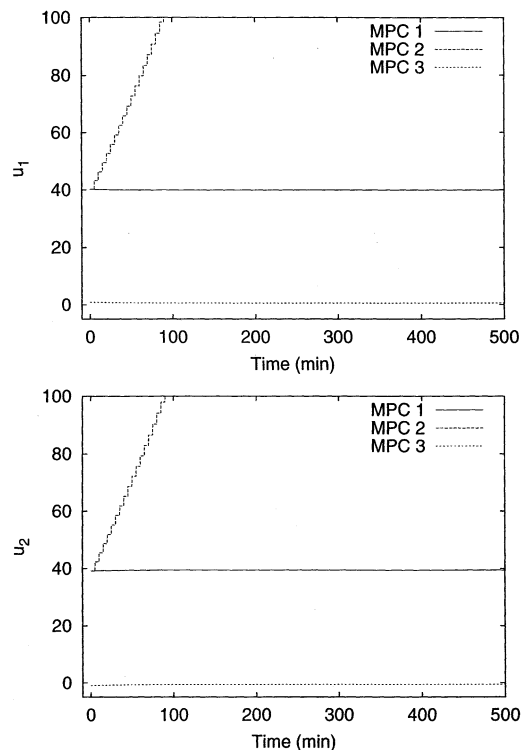


Figure 8. Ill-conditioned column inputs.
Setpoint change in the presence of gain errors.

the controlled variable. It is interesting to note that MPC 3 uses two integrating disturbances as in MPC 2, which was unstable. Therefore, closed-loop stability is not related to the number of integrators added, but to the fact that both outputs were requested to be at setpoint.

For ill-conditioned processes, the demand of offset-free performance in all outputs can lead to closed-loop instability if the model identification procedure is not sufficiently accurate. In such cases, one should relax the offset-free performance in the least important outputs to maintain closed-loop stability if the sign of the determinant of the gain matrices may be different between the model and the plant.

Conclusions

In this work the problem of designing offset-free model predictive controllers was addressed. MPC algorithms achieve offset-free performance by adding integrating disturbances to the process model (often to the controlled variables). However, the effectiveness of this procedure was proven for particular square cases only (Rawlings et al., 1994). An example was presented, in which a common and apparently reasonable choice of disturbance model leads to steady-state offset without violating detectability restrictions.

Motivated by this example, general conditions have been derived, which allow design of a proven offset-free MPC algorithm. In particular, a number of integrated disturbances equal to the number of measurements is shown to be sufficient to guarantee offset-free performance. A simple rank test is necessary and sufficient to check the suitability of the cho-

sen disturbance model. Next, it was proven that the designed controllers guarantee zero steady-state offset. These results apply to linear MPC for square and nonsquare, open-loop stable, integrating, and unstable systems. In the example, several offset-free disturbance models that add integrating disturbances to input and/or output variables were compared. It was shown that all the admissible disturbance models, that is, those with detectable augmented systems, guarantee offset-free control. The result holds if the plant follows the nonlinear model as well as the linearized model used by the controller.

These disturbance models show quite different behavior in terms of input and output transient variations. The natural and interesting question concerning the robustness properties of offset-free disturbance models is the subject of current research (Pannocchia, 2002).

A second example of an ill-conditioned distillation column was also presented to show some possible risks of demanding offset-free performance in as many variables as possible. The case of incorrect sign of the determinant of the gain matrix was considered, and it was shown that a request for zero offset in all the output variables leads to closed-loop instability. In such cases, one can relax the offset-free performance in the least important variables to maintain closed-loop stability.

After this article was submitted, we became aware of a manuscript submitted to another journal by Muske and Badgwell (MB) that treats a similar problem (Muske and Badgwell, 2002). We summarize the differences between the two articles here. Although both articles use integrating disturbances, the disturbance models are different. MB use a

block-diagonal structure for their disturbance model; we use an unstructured disturbance model. Each model choice has advantages and disadvantages, and the two are not equivalent.

We define a controlled variable, z , which may be different than the measurement, y , and determine conditions for which z can be controlled without offset. MB use y as the controlled variable as well as the measurement, and determine conditions for which y can be controlled without offset. This difference changes the focus of the two articles. MB explore the issue of how current industrial MPC algorithms can be interpreted in their article's disturbance model framework. They can therefore critique the current industrial practice of using rotation factors for integrating models, and slow rejection of disturbances when using output disturbance models. We do not discuss these issues. We show the somewhat counterintuitive result that the dimension of the disturbance model has to be as large as the measurement vector to obtain offset-free control, regardless of the number of controlled variables.

MB prove the existence of their disturbance model. The question of existence is simple in the unstructured model treated here. Existence is not as simple in the block-diagonal structures model, and MB's existence result shows that they do not lose the ability to treat any systems by choosing the block-structured model.

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Literature Cited

- Chmielewski, D., and V. Manousiouthakis, "On Constrained Infinite-Time Linear Quadratic Optimal Control," *Syst. Control Lett.*, **29**, 121 (1996).
- Clarke, D. W., and C. Mohtadi, "Properties of Generalized Predictive Control," *Automatica*, **25**, 859 (1989).
- Clarke, D. W., C. Mohtadi, and P. S. Tuffs, "Generalized Predictive Control: I. The Basic Algorithm," *Automatica*, **23**, 137 (1987a).
- Clarke, D. W., C. Mohtadi, and P. S. Tuffs, "Generalized Predictive Control: II. Extensions and Interpretations," *Automatica*, **23**, 149 (1987b).
- Cutler, C. R., and B. L. Ramaker, "Dynamic Matrix Control—A Computer Control Algorithm," AICHE Meeting, Houston, TX (1979).
- Davison, E. J., and H. W. Smith, "Pole Assignment in Linear Time-Invariant Multivariable Systems with Constant Disturbances," *Automatica*, **7**, 489 (1971).
- Davison, E. J., and H. W. Smith, "A Note on the Design of Industrial Regulators: Integral Feedback and Feedforward Controllers," *Automatica*, **10**, 329 (1974).
- Eppler, S., "Steady Offset and Integral Control in Model Predictive Control," *Advanced Independent Study*, University of Wisconsin-Madison, 1997.
- Francis, B. A., and W. M. Wonham, "The Internal Model Principle of Control Theory," *Automatica*, **12**, 457 (1976).
- Golub, G. H., and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD (1996).
- Henson, M. A., and D. E. Seborg, *Nonlinear Process Control*, Prentice Hall PTR, Upper Saddle River, NJ (1997).
- Kurtz, M. J., and M. A. Henson, "State and Disturbance Estimation

- for Nonlinear Systems Affine in the Unmeasured Variables," *Comput. Chem. Eng.*, **22**, 1441 (1998).
- Kwakernaak, H., and R. Sivan, *Linear Optimal Control Systems*, Wiley, New York (1972).
- Kwon, W. H., and D. G. Byun, "Receding Horizon Tracking Control as a Predictive Control and its Stability Properties," *Int. J. Control*, **50**, 1807 (1989).
- Lee, J. H., M. S. Gelormino, and M. Morari, "Model Predictive Control of Multi-Rate Sampled-Data Systems: A State-Space Approach," *Int. J. Control*, **55**, 153 (1992).
- Lee, J. H., M. Morari, and C. E. Garcia, "State-Space Interpretation of Model Predictive Control," *Automatica*, **30**, 707 (1994).
- Lundstrom, P., J. H. Lee, M. Morari, and S. Skogestad, "Limitations of Dynamic Matrix Control," *Comput. Chem. Eng.*, **19**, 409 (1995).
- Marquis, P., and J. P. Broustail, "SMOC, A Bridge Between State Space and Model Predictive Controllers: Application to the Automation of a Hydrotreating Unit," *Proceedings of the 1988 IFAC Workshop on Model Based Process Control*, T. J. McAvoy, Y. Arkun, and E. Zafiriou, eds., Pergamon Press, Oxford, p. 37 (1988).
- Morari, M., and G. Stephanopoulos, "Minimizing Unobservability in Inferential Control Schemes," *Int. J. Control*, **31**, 367 (1980).
- Morari, M., and E. Zafiriou, *Robust Process Control*, Prentice Hall, Englewood Cliffs, NJ (1989).
- Muske, K. R., and T. A. Badgwell, "Disturbance Modeling for Offset-Free Linear Model Predictive Control," *J. Process Control*, **12**, 617 (2002).
- Muske, K. R., and J. B. Rawlings, "Model Predictive Control with Linear Models," *AIChE J.*, **39**, 262 (1993).
- Pannocchia, G., "Robust Offset-Free Model Predictive Control," IFAC 2002 World Congress, Barcelona, Spain (2002).
- Qiu, L., and E. Davison, "Performance Limitations of Non-Minimum Phase Systems in the Servomechanism Problem," *Automatica*, **29**, 337 (1993).
- Rawlings, J. B., E. S. Meadows, and K. R. Muske, "Nonlinear Model Predictive Control: A Tutorial and Survey," *ADCHEM Proc.*, Kyoto, Japan, p. 185 (1994).
- Rawlings, J. B., and K. R. Muske, "Stability of Constrained Receding Horizon Control," *IEEE Trans. Autom. Control*, **AC-38**, 1512 (1993).
- Richalet, J., A. Rault, J. L. Testud, and J. Papon, "Model Predictive Heuristic Control: Applications to Industrial Processes," *Automatica*, **14**, 413 (1978).
- Scokaert, P. O. M., and J. B. Rawlings, "Constrained Linear Quadratic Regulation," *IEEE Trans. Autom. Control*, **AC-43**, 1163 (1998).
- Skogestad, S., and I. Postlethwaite, *Multivariable Feedback Control, Analysis and Design*, Wiley, New York (1996).
- Smith, H. W., and E. J. Davison, "Design of Industrial Regulators. Integral Feedback and Feedforward Control," *Proc. IEE*, **119**, 1210 (1972).
- Sontag, E. D., *Mathematical Control Theory*, 2nd ed., Springer-Verlag, New York (1998).

Appendix: Proofs

Proof of Lemma 2

Since the estimator matrix $\tilde{A} - \tilde{L}\tilde{C}$ is stable and the input and the plant have reached steady values, we have from Eq. 14 that the state and the disturbance estimates also reach steady values. We denote with x_s^- and d_s^- the steady-state values of the model state and disturbance, respectively, prior to filtering. We also denote with x_s^+ and d_s^+ the corresponding steady-state values after filtering (in general, they can be different). Let e_s be the steady-state output reconstruction error defined as

$$e_s = y_s - Cx_s^- - C_d d_s^-.$$

From Eqs. 14 and 15 we have

$$x_s^- = Ax_s^- + Bu_s + B_d d_s^- + AL_x e_s \quad (\text{A1a})$$

$$0 = L_d e_s. \quad (\text{A1b})$$

From the target calculation Eq. 18b, we have

$$x_t = Ax_t + Bu_t + B_d d_s^-, \quad (\text{A2})$$

in which we used the fact that $d_s^+ = d_s^- + L_d e_s = d_s^-$. Moreover, since constraints are not active at steady state and infinite horizon is used, the steady-state input, u_s , is given by

$$u_s = u_t + K(x_s^+ - x_t) = u_t + K(x_s^- + L_x e_s - x_t). \quad (\text{A3})$$

Combining Eqs. A1a, A2, and A3, we obtain

$$x_s^- - x_t = (I - A - BK)^{-1}(A + BK)L_x e_s, \quad (\text{A4})$$

in which $(I - A - BK)^{-1}$ exists since $A + BK$ is a strictly stable matrix (see, e.g., Kwakernaak and Sivan, 1972). Using Eqs. 18b and A4, we obtain

$$\begin{aligned} Hy_s - \bar{z} &= H(e_s + Cx_s^- + C_d d_s^- - Cx_t - C_d d_s^-) \\ &= H[I + C(I - A - BK)^{-1}(A + BK)L_x]e_s \\ &= K_e e_s. \end{aligned}$$

Since Eq. A1b we have that $e_s \in \text{null}(L_d)$, and from Eq. 25 we have that $K_e e_s = 0$, and therefore Eq. 26.

Proof of Lemma 3

The first statement (Eq. 28) follows immediately from Eq. 27. From Eq. 15 we have that $L_d = L_2$, and we prove that $\text{rank } L_2 = p$ by contradiction. Under the assumptions of Lemma 1, the augmented system is detectable. Hence, a filter gain, \tilde{L} , exists such that the estimator characteristic closed-loop matrix $\tilde{A} - \tilde{L}\tilde{C}$ is stable. $L_2 \in \mathbb{R}^{p \times p}$ can be rewritten using the Schur decomposition [see, e.g., Golub and Van Loan (1996, p. 313)]

$$L_2 = \Gamma \Lambda \Gamma^H, \quad (\text{A5})$$

in which $\Gamma \in \mathbb{C}^{p \times p}$ is a unitary matrix ($\Gamma \Gamma^H = \Gamma^H \Gamma = I$) and $\Lambda \in \mathbb{C}^{p \times p}$ is an upper triangular matrix whose diagonal con-

tains the eigenvalues λ_i of the matrix L_2 . Moreover, Γ can be chosen in such a way that the eigenvalues appear in any order along the diagonal.

Suppose L_2 is singular (or equivalently not full rank), we can choose Γ such that the last eigenvalue λ_p is zero. Thus, the elements of the last row of Λ are all equal to zero. Then, we have

$$\begin{aligned} \tilde{A} - \tilde{L}\tilde{C} &= \begin{bmatrix} A - L_1 C & B_d - L_1 C_d \\ -\Gamma \Lambda \Gamma^H C & I - \Gamma \Lambda \Gamma^H C_d \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} A - L_1 C & (B_d - L_1 C_d)\Gamma \\ -\Lambda \Gamma^H C & I - \Lambda \Gamma^H C_d \Gamma \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Gamma^H \end{bmatrix}. \end{aligned}$$

Since $\begin{bmatrix} I & 0 \\ 0 & \Gamma \end{bmatrix}$ is unitary, the eigenvalues of $\tilde{A} - \tilde{L}\tilde{C}$ are equal to the eigenvalues of

$$\hat{A} = \begin{bmatrix} A - L_1 C & (B_d - L_1 C_d)\Gamma \\ -\Lambda \Gamma^H C & I - \Lambda \Gamma^H C_d \Gamma \end{bmatrix}.$$

Since the last row of Λ has only null elements, the matrix \hat{A} has the following form:

$$\hat{A} = \begin{bmatrix} \times & \dots & \dots & \times \\ \vdots & & & \vdots \\ \times & \dots & \dots & \times \\ 0 & \dots & \dots & 1 \end{bmatrix}. \quad (\text{A6})$$

Therefore, 1 is an eigenvalue of \hat{A} and of $\tilde{A} - \tilde{L}\tilde{C}$, which contradicts the stability of $\tilde{A} - \tilde{L}\tilde{C}$. Hence, $L_d = L_2$ cannot be rank deficient.

Proof of Theorem 1

Since the assumptions of Lemma 3 are satisfied, we have that L_d is square and full rank. Therefore, the null space of L_d is only the zero vector. Thus, the assumptions of Lemma 2 are satisfied and there is zero offset in the controlled variables.

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